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Abstract

Given a set of flight legs to be flown by a single type of aircraft, the simultaneous aircraft routing and crew scheduling problem consists of determining a minimum-cost set of aircraft routes and crew pairings such that each flight leg is covered by one aircraft and one crew, and side constraints are satisfied. While some side constraints such as maximum flight time and maintenance requirements involve only crews or aircraft, linking constraints impose minimum connection times for crews that depend on aircraft connections. To handle these linking constraints, a solution approach based on Benders decomposition is proposed. The solution process iterates between a master problem which solves the aircraft routing problem, and a subproblem which solves the crew pairing problem. Because of their particular structure, both of these problems are themselves decomposed and solved by column generation. A heuristic branch-and-bound method is used to compute integer solutions. On a set of test instances based on data provided by an airline, the integrated approach produced significant cost savings in comparison with the sequential planning process commonly used in practice. The largest instance solved contains more than 500 flight legs over a three-day period.

Keywords: air transportation; integer programming; multi-commodity network flow; Benders decomposition; Dantzig-Wolfe decomposition; column generation.

Résumé

Étant donné un ensemble de vols devant être couverts par un seul type d'avions, le problème simultané de routage d'avions et d'horaires d'équipages consiste à déterminer un ensemble d'itinéraires d'avion et de rotations d'équipage tels que chaque vol est couvert par un appareil et un équipage, tout en satisfaisant un ensemble de contraintes supplémentaires. Alors que certaines contraintes supplémentaires telles que le temps maximum de vol et les contraintes d'entretien ne concernent que les équipages ou les avions, des contraintes liantes imposent des temps minimums de connexion pour les équipages qui dépendent des connexions utilisées par les appareils. Pour traiter ces contraintes liantes, nous proposons une approche de résolution basée sur la décomposition de Benders. Le processus de résolution itère entre un problème maître qui résout le problème des itinéraires d'avions, et un sous-problème qui résout le problème des rotations d'équipages. En raison de leur structure particulière, ces problèmes sont eux-mêmes décomposés et résolus par génération de colonnes. Une méthode d'évaluation et de séparation heuristique est utilisée afin d'obtenir des solutions entières. Sur un ensemble d'instances basées sur des données réelles fournies par un transporteur, l'approche intégrée a généré des économies importantes par rapport au processus de planification séquentiel souvent utilisé en pratique. La plus grande instance résolue comporte plus de 500 vols sur une période de 3 jours.

1 Introduction

After creating a schedule that defines origin and destination cities as well as departure and arrival times for each flight leg to be flown during a given period, most airlines use a sequential procedure to plan their operations. The first step of this procedure is the *fleet assignment problem* which consists of assigning an aircraft type to each flight leg so as to maximize anticipated profits. For each aircraft type, an *aircraft routing problem* is then solved to determine the sequence of flight legs to be flown by each individual aircraft so as to cover each leg exactly once while ensuring appropriate aircraft maintenance. Given fixed aircraft routes and a set of work rules usually defined by a collective agreement, the airline then builds crew rotations or *pairings* by solving a *crew scheduling problem*. In the last step, pairings are finally combined to form personalized monthly schedules that are assigned to employees by solving a *crew bidding* or *crew rostering problem*.

Using a sequential procedure considerably reduces the complexity of the planning process but may also yield solutions that are far from optimal. Furthermore, finding feasible solutions may become difficult when flexibility is reduced by previously made decisions. This paper introduces a model and a solution methodology for the simultaneous routing of aircraft and scheduling of crews. It thus represents an attempt at integrating two interacting parts of the planning process. The main source of interaction between aircraft routing and crew scheduling resides in variable connection times. In general terms, a crew pairing is a sequence of duty and rest periods that typically lasts between two and five days. Every pairing begins and ends at a specific location, called the crewbase, and must satisfy a set of applicable work rules related to a large number of factors such as flight time, rest time, connection time, etc. But because the minimum connection time required between two successive flight legs covered by the same crew depends on whether the same aircraft is used on both legs, the set of feasible pairings depends on the aircraft routing decisions made in the previous step. Hence, a suboptimal solution is likely to be obtained if a sequential planning procedure is used. The main contribution of this paper is to present a model and a solution approach for handling both types of decisions simultaneously so as to obtain an improved global solution. This model would typically be solved monthly before preparing personalized crew schedules for the next month. It can also be used at the operational level to update aircraft routes and crew pairings following schedule perturbations (Stojković, 1999).

The operations research literature contains few references regarding the integration of aircraft routing and crew scheduling decisions. An effort in this direction is the work of Barnhart et al. (1998b) who proposed an integrated but approximate model for combined fleet assignment and crew scheduling. The approach does not truly solve the two problems simultaneously but rather incorporates a relaxation of the crew scheduling problem in the fleet assignment model. This relaxation, which is based on a duty network (Barnhart and Shenoi, 1998), ensures that all flight legs are covered by eligible crews, but does not impose constraints on the maximum number of duties within a pairing or the maximum time away from the crewbase. Very recently, Klabjan (1999) described an approach that solves the crew scheduling and aircraft routing problems sequentially but adds plane count constraints to the crew scheduling model so as to ensure the feasibility of the resulting aircraft routing problem. In addition, this model allows the departure time of each flight leg to be moved within a given time window so as to further reduce crew costs. On test instances involving up to 450 flight legs, the method produced very significant savings. Issues related to the introduction of maintenance and crew considerations in the fleet assignment problem were also discussed by Clarke et al. (1996). Finally, other interesting contributions with respect to the integration of the planning process are the two approaches presented by Barnhart et al. (1998a) and Desaulniers et al. (1997b) for the combined fleet assignment and aircraft routing problem.

While few integrated planning models exist, several modeling and solution approaches have been proposed to address the individual components of the sequential planning process described above (Yu, 1998). Models for the fleet assignment problem were proposed, among others, by Abara (1989) and Hane et al. (1995), while the aircraft routing problem was addressed by Feo and Bard (1989), Clarke et al. (1997), Talluri (1998) and Gopalan and Talluri (1998). Numerous contributions regarding the different variants of the crew scheduling problem can also be found in the operations research literature. In particular, column generation was first applied to the crew pairing problem by Lavoie et al. (1988), while further developments and alternative solution methods can be found in the recent works of Hoffman and Padberg (1993), Graves et al. (1993), Barnhart et al. (1995), Vance et al. (1997) and Desaulniers et al. (1997a).

In this paper, we assume that the fleet assignment problem has been solved so that the type of aircraft assigned to each flight leg is known. For ease of exposition, we also assume that crews are qualified to fly a single type of aircraft, although this assumption is easily relaxed. In this context, the simultaneous aircraft routing and crew scheduling problem decomposes into one problem for each aircraft type. Given a set of flight legs to be flown by the aircraft of a specific type, the problem is to determine a minimum-cost set of aircraft routes and crew pairings such that each flight leg is covered by one aircraft and one crew, and side constraints are satisfied. While some side constraints such as maximum flight time and maintenance requirements involve only crews or aircraft, linking constraints impose minimum connection times for crews that depend on aircraft connections. To handle these complicating constraints, we introduce a solution approach based on mathematical decomposition that takes advantage of the particular structure of the problem.

The remainder of the article is organized as follows. The next section introduces some notation and a mathematical formulation of the problem while Section 3 presents the solution methodology. Computational experiments that show the benefits of integration are reported in Section 4. Conclusions and directions for future work are discussed in the final section.

2 Mathematical Formulation

Consider a set L of flight legs to be flown by a single aircraft type during the planning horizon. Each flight leg $l \in L$ is defined by origin and destination stations, and fixed departure and arrival dates and times. Let $G = (N, A)$ be a time-space network where N

is the node set and A is the arc set. Each node $i \in N$ corresponds to a flight leg $l_i \in L$ that must be covered exactly once by one aircraft and one crew. Each arc $(i, j) \in A$ represents a feasible connection between two successive flight legs: an arc is defined between nodes i and j if the destination station of leg l_i is the departure station of leg l_j and if the connection time between the two legs is larger than a given station-specific threshold that represents the minimum connection time when both legs are covered by the same aircraft.

Let F and K denote the sets of available aircraft and crews, respectively. For each aircraft $f \in F$, let o^f and d^f be nodes that represent, respectively, the origin of this aircraft at the beginning of the planning horizon and its destination at the end. Define $O^f \subseteq \{(o^f, j)|j \in N\}$ as the set of arcs linking the origin node of aircraft f to nodes representing legs that can be covered first by this aircraft at the beginning of the horizon. The set $D^f \subseteq \{(i, d^f)|i \in N\}$ is defined similarly for legs that can be covered last at the end of the planning horizon. One then defines a network $G^f = (N^f, A^f)$ where $N^f = N \cup \{o^f, d^f\}$ and $A^f = A \cup O^f \cup D^f$. Sets O^f and D^f can be used to impose initial and final conditions on each aircraft. If no particular condition is to be imposed on aircraft f, then arcs (o^f, i) and (i, d^f) can be defined for every node $i \in N$. For every crew $k \in K$, nodes o^k and d^k, sets O^k and D^k, and a network $G^k = (N^k, A^k)$ are defined in a similar fashion.

A solution to the aircraft routing and crew scheduling problem can be expressed as a collection of paths: to each aircraft and each crew is associated a sequence of arcs from its origin node to its destination node. Hence, for each aircraft $f \in F$ and each arc $(i, j) \in A^f$, define a binary variable X^f_{ij} with cost c^f_{ij} taking value 1 if the arc is used by aircraft f, and 0 otherwise. Corresponding binary variables Y^k_{ij} with cost c^k_{ij} are defined for each crew $k \in K$ and each arc $(i, j) \in A^k$.

These paths in the graphs G^f and G^k must usually satisfy a set of operational constraints which can be modeled in terms of resource consumption. For instance, aircraft paths must satisfy maintenance requirements while crew paths must respect daily limits on total work time and total flight time. For each aircraft $f \in F$, let R^f be the set of relevant resources. For each node $i \in N^f$ and each resource $r \in R^f$, a resource variable T^{fr}_i is then associated with aircraft f. This variable represents the cumulative consumption of resource r up to node i on the path for aircraft f. A resource consumption t^{fr}_{ij} is then associated with each arc $(i, j) \in A^f$ and each resource $r \in R^f$. Finally, constraints on total resource consumption along a path are imposed by restricting each T^{fr}_i variable to lie in the interval $[a^{fr}_i, b^{fr}_i]$. If node i does not belong to the path for aircraft f, then $T^{fr}_i = 0$. Equivalent variables T^{kr}_i, and parameters t^{kr}_{ij}, a^{kr}_i and b^{kr}_i are defined for each crew $k \in K$ and resource $r \in R^k$.

Given two flight legs $l_i, l_j \in L$ such that $(i, j) \in A$, the connection between these legs is said to be *short* if the difference between the departure time of leg l_j and the arrival time of leg l_i is smaller than a given threshold. In this case, legs l_i and l_j can be covered by the same crew only if both legs are covered by the same aircraft. Otherwise, insufficient time will be available for the crew to make the connection. Let $C \subseteq A$ be the set of arcs representing short connections in the network G.

A formulation for the simultaneous aircraft routing and crew scheduling problem, which is a special case of the unified framework for vehicle routing and crew scheduling described by Desaulniers et al. (1998), can be stated as follows:

$$\text{Minimize} \quad \sum_{f\in F} \sum_{(i,j)\in A^f} c_{ij} X_{ij}^f + \sum_{k\in K} \sum_{(i,j)\in A^k} c_{ij} Y_{ij}^k \tag{1}$$

subject to

$$\sum_{f\in F} \sum_{j:(i,j)\in A^f} X_{ij}^f = 1 \qquad (i \in N) \tag{2}$$

$$\sum_{k\in K} \sum_{j:(i,j)\in A^k} Y_{ij}^k = 1 \qquad (i \in N) \tag{3}$$

$$\sum_{k\in K} Y_{ij}^k - \sum_{f\in F} X_{ij}^f \leq 0 \qquad ((i,j) \in C) \tag{4}$$

$$\sum_{j:(o^f,j)\in A^f} X_{o^f j}^f = 1 \qquad (f \in F) \tag{5}$$

$$\sum_{j:(i,j)\in A^f} X_{ij}^f - \sum_{j:(j,i)\in A^f} X_{ji}^f = 0 \qquad (f \in F; i \in N) \tag{6}$$

$$X_{ij}^f (T_i^{fr} + t_{ij}^{fr} - T_j^{fr}) \leq 0 \qquad (f \in F; (i,j) \in A^f; r \in R^f) \tag{7}$$

$$a_i^{fr} (\sum_{j:(i,j)\in A^f} X_{ij}^f) \leq T_i^{fr} \leq b_i^{fr} (\sum_{j:(i,j)\in A^f} X_{ij}^f) \qquad (f \in F; i \in N^f; r \in R^f) \tag{8}$$

$$X_{ij}^f \in \{0,1\} \qquad (f \in F; (i,j) \in A^f) \tag{9}$$

$$\sum_{j:(o^k,j)\in A^k} Y_{o^k j}^k = 1 \qquad (k \in K) \tag{10}$$

$$\sum_{j:(i,j)\in A^k} Y_{ij}^k - \sum_{j:(j,i)\in A^k} Y_{ji}^k = 0 \qquad (k \in K; i \in N) \tag{11}$$

$$Y_{ij}^k (T_i^{kr} + t_{ij}^{kr} - T_j^{kr}) \leq 0 \qquad (k \in K; (i,j) \in A^k; r \in R^k) \tag{12}$$

$$a_i^{kr} (\sum_{j:(i,j)\in A^k} Y_{ij}^k) \leq T_i^{kr} \leq b_i^{kr} (\sum_{j:(i,j)\in A^k} Y_{ij}^k) \qquad (k \in K; i \in N^k; r \in R^k) \tag{13}$$

$$Y_{ij}^k \in \{0,1\} \qquad (k \in K; (i,j) \in A^k). \tag{14}$$

The objective function (1) minimizes the sum of all costs associated with the connection arcs. Constraints (2) and (3) ensure that each leg is covered by exactly one aircraft and one crew while constraints (4) guarantee that a crew does not change aircraft when the connection time is too short. Constraints (5) state that a path must be assigned to each aircraft. Flow conservation along these paths is enforced through (6). Relations (7) define the resource extension process on arcs of the networks: the cumulative consumption of resource r at node j must be larger than or equal to the cumulative consumption at node i plus the amount consumed on the arc (i, j) whenever $X_{ij}^{f} = 1$. These resource variables are restricted to a set of feasible values by constraints (8). Constraints (10)-(13) serve the same purpose for crews. Finally, all flow variables are restricted to be binary by (9) and (14).

For each aircraft, a resource is used to represent the number of days since the last maintenance. Setting the upper bound b_i^{fr} equal to ℓ ensures that a maintenance opportunity is provided at least once every ℓ days. By using a negative resource consumption of $-\ell$ units, this resource is reset to zero on arcs associated with a connection in a maintenance station, provided that the connection time is sufficiently long for maintenance to be performed. For crews, three resources are used to define feasible duties: flight time, duty time and number of landings. A fourth resource is used to represent the number of days since the beginning of the pairing. When the duration of a connection at the crewbase exceeds a certain number of days, the value of this resource is reset to zero. Further details concerning the use of resources in network modeling can be found in the work of Desaulniers et al. (1999).

Given the complexity of collective agreements, evaluating the exact cost of a crew pairing is often difficult to accomplish within a mathematical model. However, because each flight leg must be covered by exactly one crew, a large portion of total crew costs is in fact a fixed expense. Reasonable approximations of variable expenses can then be obtained by assigning appropriate cost coefficients to the connection arcs. In our model, two main types of costs are considered: (a) if a rest period between two consecutive duties does not take place at the crewbase, a fixed cost is imposed to account for accommodation; (b) if the duration of a connection within a duty exceeds a given threshold, a cost proportional to the connection time is imposed to account for paid waiting time and poor schedule quality. In some situations, allowing extra crews to travel as passengers on certain flights may reduce the total cost of the schedule. This possibility is introduced in the model by replacing each flight node $i \in N$ by a departure node i_d and an arrival node i_a. These two nodes are then linked by a regular flight arc as well as a *deadhead* arc (with adequate cost and resource consumptions) that represents the travel of a crew on this flight.

Finally, crews are sometimes qualified to fly different types of aircraft that belong to the same aircraft family. Because fleet assignment decisions are assumed to be made separately, model (1)-(14) adapts easily to this situation by defining subsets $F_i \subseteq F$ of aircraft that are compatible with each leg $l_i \in L$, and adjusting constraints (2) and (4) accordingly. The networks G^f should then be customized for each particular type of aircraft.

3 Solution Methodology

Model (1)-(14) presents two separate components that are linked only by constraints (4). A technique to take advantage of this particular structure consists of applying Benders decomposition (Benders, 1962) to the model. However, this would require dualizing the complex set of constraints (10)-(13). Instead, we reformulate model (1)-(14) by using the Dantzig-Wolfe decomposition principle (Dantzig and Wolfe, 1960), and then show how this new formulation can itself be decomposed by using Benders decomposition.

3.1 Dantzig-Wolfe Decomposition

Since the objective function (1) and constraints (5)-(14) are separable by aircraft or crew, model (1)-(14) has a block-angular structure with linking constraints (2)-(4). Constraints (5)-(9) define a path structure for each aircraft $f \in F$ while constraints (10)-(14) define a similar path structure for each crew $k \in K$. For a given $f \in F$, let Ω^f be the set of extreme points of the convex hull of (5)-(9). Each extreme point $\omega \in \Omega^f$ can be described by a binary flow vector $\boldsymbol{x}_\omega = (x_{ij\omega} | (i,j) \in A^f)$ and a resource vector $\boldsymbol{\tau}_\omega = (\tau^r_{i\omega} | i \in N^f; r \in R^f)$. Also, if Ω^k denotes the set of extreme points of the convex hull of (10)-(14) for a given crew $k \in K$, then each extreme point $\omega \in \Omega^k$ can be described by vectors $\boldsymbol{y}_\omega = (y_{ij\omega} | (i,j) \in A^f)$ and $\boldsymbol{\tau}_\omega = (\tau^r_{i\omega} | i \in N^k; r \in R^k)$. Assigning a non-negative variable θ_ω to each extreme point ω, any solution in the original variables X^f_{ij}, T^{fr}_i, Y^k_{ij} and T^{kr}_i satisfying (5)-(14) can thus be expressed as non-negative convex combinations of these extreme points:

$$X^f_{ij} = \sum_{\omega \in \Omega^f} x_{ij\omega} \theta_\omega \qquad (f \in F; (i,j) \in A^f) \tag{15}$$

$$T^{fr}_i = \sum_{\omega \in \Omega^f} \tau^r_{i\omega} \theta_\omega \qquad (f \in F; i \in N^f; r \in R^f) \tag{16}$$

$$Y^k_{ij} = \sum_{\omega \in \Omega^k} y_{ij\omega} \theta_\omega \qquad (k \in K; (i,j) \in A^k) \tag{17}$$

$$T^{kr}_i = \sum_{\omega \in \Omega^k} \tau^r_{i\omega} \theta_\omega \qquad (k \in K; i \in N^k; r \in R^k) \tag{18}$$

$$\sum_{\omega \in \Omega^f} \theta_\omega = 1 \qquad (f \in F) \tag{19}$$

$$\sum_{\omega \in \Omega^k} \theta_\omega = 1 \qquad (k \in K) \tag{20}$$

$$\theta_\omega \geq 0 \qquad (f \in F; \omega \in \Omega^f) \tag{21}$$

$$\theta_\omega \geq 0 \qquad (k \in K; \omega \in \Omega^k). \tag{22}$$

For a given aircraft $f \in F$, the elements of Ω^f represent paths between the origin node o^f and the destination node d^f of G^f. Hence, for every extreme point $\omega \in \Omega^f$, define binary constants a^i_ω and b^{ij}_ω that take value 1 if node $i \in N^f$ and arc $(i,j) \in A^f$ belong to this path, respectively. Let also c_ω be the cost of sending one unit of flow between o^f and d^f along path ω. Defining equivalent notation for every crew $k \in K$, the original model (1)-(14) can be rewritten as the following integer master problem:

$$\text{Minimize} \quad \sum_{f \in F} \sum_{\omega \in \Omega^f} c_\omega \theta_\omega + \sum_{k \in K} \sum_{\omega \in \Omega^k} c_\omega \theta_\omega \tag{23}$$

subject to

$$\sum_{f \in F} \sum_{\omega \in \Omega^f} a^i_\omega \theta_\omega = 1 \qquad (i \in N) \tag{24}$$

$$\sum_{k \in K} \sum_{\omega \in \Omega^k} a^i_\omega \theta_\omega = 1 \qquad (i \in N) \tag{25}$$

$$\sum_{k \in K} \sum_{\omega \in \Omega^k} b^{ij}_\omega \theta_\omega - \sum_{f \in F} \sum_{\omega \in \Omega^f} b^{ij}_\omega \theta_\omega \leq 0 \qquad ((i,j) \in C) \tag{26}$$

$$\sum_{\omega \in \Omega^f} \theta_\omega = 1 \qquad (f \in F) \tag{27}$$

$$\sum_{\omega \in \Omega^k} \theta_\omega = 1 \qquad (k \in K) \tag{28}$$

$$\theta_\omega \in \{0,1\} \qquad (f \in F; \omega \in \Omega^f) \tag{29}$$

$$\theta_\omega \in \{0,1\} \qquad (k \in K; \omega \in \Omega^k). \tag{30}$$

Constraints (24)-(26) are equivalent to constraints (2)-(4) while constraints (27) and (28) correspond to the path structures defined by (5)-(8) and (10)-(13), respectively. Finally, as explained by Desaulniers et al. (1998), binary requirements on path variables imposed by (29) and (30) are equivalent to the integrality requirements on the original binary arc flow variables imposed by (9) and (14)

Although the integer master problem (23)-(30) is equivalent to model (1)-(14), the large size of the sets Ω^f and Ω^k prohibits a complete enumeration of all aircraft and crew path variables. However, this model can be solved by a branch-and-bound algorithm in which lower bounds are computed by solving linear relaxations with a column generation approach. At each iteration of the column generation process, a restricted master problem is solved with only a subset of all variables. This problem is obtained by replacing the sets Ω^f and Ω^k by the subsets $\Omega^f_t \subseteq \Omega^f$ and $\Omega^k_t \subseteq \Omega^k$ of extreme points available at iteration $t = 0, 1, \ldots$ New variables for the master problem are then generated at each iteration by solving a resource-constrained shortest-path problem for each network G^f $(f \in F)$ and G^k $(k \in K)$. Extreme points can indeed be identified by sending one unit of flow from the source to the sink of the network. At each iteration, arc costs are modified to reflect the current values of the dual variables associated with the constraints of the restricted master problem. The column generation process stops when no negative-cost path can be identified in any network.

Let $\boldsymbol{\alpha} = (\alpha_i | i \in N)$, $\boldsymbol{\beta} = (\beta_i | i \in N)$, $\boldsymbol{\gamma} = (\gamma_{ij} \leq 0 | (i,j) \in C)$, $\boldsymbol{\delta} = (\delta_f | f \in F)$ and $\boldsymbol{\zeta} = (\zeta_k | k \in K)$ be the dual variables associated with constraint sets (24)-(28), respectively. The subproblem for aircraft $f \in F$ is to minimize the objective

$$\sum_{(i,j)\in A^f} c_{ij} X_{ij}^f - \sum_{j:(i,j)\in A^f} \alpha_i X_{ij}^f + \sum_{(i,j)\in C} \gamma_{ij} X_{ij}^f - \delta^f \tag{31}$$

under constraints (5)-(9). For crew $k \in K$, the subproblem is to minimize the objective

$$\sum_{(i,j)\in A^k} c_{ij} Y_{ij}^k - \sum_{j:(i,j)\in A^k} \beta_i Y_{ij}^k - \sum_{(i,j)\in C} \gamma_{ij} Y_{ij}^k - \zeta^k \tag{32}$$

under constraints (10)-(14). These subproblems are in fact shortest path problems with resource variables and can be solved by dynamic programming (Desrochers, 1986). Every subproblem whose solution has a negative cost provides a column which is then added to the restricted master problem.

To compute an optimal integer solution of model (23)-(30), different branching schemes can be devised. For example, branching decisions can be made on the original X_{ij}^f and Y_{ij}^k variables. A more simple scheme consists of branching directly on the flow variables θ_ω. However, because imposing the constraint $\theta_\omega = 0$ is difficult to accomplish when using column generation, one must often be content with finding heuristic solutions by a depth-only search. In this case, a fractional variable θ_ω is selected at each node of the tree and the constraint $\theta_\omega = 1$ is added to the restricted master problem. This process is repeated until all variables satisfy integrality requirements or the restricted master problem becomes infeasible. When several variables take fractional values that exceed a certain threshold, branching decisions can also be made simultaneously on all variables to accelerate the search.

3.2 Benders Decomposition

When the number of flight legs and the number of short connections each exceed a few hundreds, solving model (23)-(30) by column generation becomes difficult because of the large number of constraints in the restricted master problem. However, this model can itself be decomposed so as to obtain a pair of problems that can be solved more easily. We first show how the LP relaxation of (23)-(30) can be solved by Benders decomposition and then explain how this approach can be embedded in a branch-and-bound algorithm to obtain integer solutions.

For given non-negative values $\bar{\theta}_\omega$ $(f \in F; \omega \in \Omega^f)$ satisfying constraints (24) and (27), the LP relaxation of model (23)-(30) reduces to the following *primal subproblem*:

$$\text{Minimize} \quad \sum_{k\in K} \sum_{\omega\in\Omega^k} c_\omega \theta_\omega \tag{33}$$

subject to

$$\sum_{k\in K}\sum_{\omega\in\Omega^k} a^i_\omega\theta_\omega = 1 \qquad (i\in N) \tag{34}$$

$$\sum_{k\in K}\sum_{\omega\in\Omega^k} b^{ij}_\omega\theta_\omega \le \sum_{f\in F}\sum_{\omega\in\Omega^f} b^{ij}_\omega\bar{\theta}_\omega \qquad ((i,j)\in C) \tag{35}$$

$$\sum_{\omega\in\Omega^k}\theta_\omega = 1 \qquad (k\in K) \tag{36}$$

$$\theta_\omega \ge 0 \qquad (k\in K;\omega\in\Omega^k). \tag{37}$$

Again, let $\boldsymbol{\beta} = (\beta_i|i\in N)$, $\boldsymbol{\gamma} = (\gamma_{ij}\le 0|(i,j)\in C)$ and $\boldsymbol{\zeta} = (\zeta_k|k\in K)$ be the dual variables associated with constraints (34), (35) and (36), respectively. The dual of (33)-(37) is the following *dual subproblem*:

$$\text{Maximize} \quad \sum_{i\in N}\beta_i + \sum_{(i,j)\in C}\sum_{f\in F}\sum_{\omega\in\Omega^f} b^{ij}_\omega\bar{\theta}_\omega\gamma_{ij} + \sum_{k\in K}\zeta_k \tag{38}$$

subject to

$$\sum_{i\in N} a^i_\omega\beta_i + \sum_{(i,j)\in C} b^{ij}_\omega\gamma_{ij} + \zeta_k \le c_\omega \qquad (k\in K;\omega\in\Omega^k) \tag{39}$$

$$\gamma_{ij}\le 0 \qquad ((i,j)\in C). \tag{40}$$

Assuming that $c_\omega \ge 0$ for all $k\in K$ and $\omega\in\Omega^k$, the dual subproblem is always feasible since the null vector $\mathbf{0}$ satisfies constraints (39) and (40). Hence, the primal subproblem is either infeasible or it is feasible and bounded. Let $\boldsymbol{\Delta}$ denote the polyhedron defined by constraints (39) and (40), and let $P_{\boldsymbol{\Delta}}$ and $R_{\boldsymbol{\Delta}}$ be the sets of extreme points and extreme rays of $\boldsymbol{\Delta}$, respectively.

If, for given non-negative values $\bar{\theta}_\omega$ $(f\in F;\omega\in\Omega^f)$ satisfying constraints (24) and (27), the inequality

$$\sum_{i\in N}\beta_i + \sum_{(i,j)\in C}\sum_{f\in F}\sum_{\omega\in\Omega^f} b^{ij}_\omega\bar{\theta}_\omega\gamma_{ij} + \sum_{k\in K}\zeta_k \le 0 \tag{41}$$

is satisfied for every extreme ray $(\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\zeta})\in R_{\boldsymbol{\Delta}}$, then the dual subproblem is bounded and the primal subproblem is feasible. The optimal value of both subproblems is then equal to

$$\max_{(\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\zeta})\in P_{\boldsymbol{\Delta}}} \sum_{i\in N}\beta_i + \sum_{(i,j)\in C}\sum_{f\in F}\sum_{\omega\in\Omega^f} b^{ij}_\omega\bar{\theta}_\omega\gamma_{ij} + \sum_{k\in K}\zeta_k. \tag{42}$$

Otherwise, there is an extreme ray $(\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\zeta})\in R_{\boldsymbol{\Delta}}$ for which the dual subproblem is unbounded and the primal subproblem is infeasible.

Introducing the additional free variable z_0, the LP relaxation of model (23)-(30) can thus be reformulated as the following *Benders master problem*:

$$\text{Minimize} \quad \sum_{f \in f} \sum_{\omega \in \Omega^f} c_\omega \theta_\omega + z_0 \tag{43}$$

subject to

$$z_0 - \sum_{f \in F} \sum_{\omega \in \Omega^f} \sum_{(i,j) \in C} b_\omega^{ij} \gamma_{ij} \theta_\omega \geq \sum_{i \in N} \beta_i + \sum_{k \in K} \zeta_k \qquad ((\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\zeta}) \in P_{\boldsymbol{\Delta}}) \tag{44}$$

$$- \sum_{f \in F} \sum_{\omega \in \Omega^f} \sum_{(i,j) \in C} b_\omega^{ij} \gamma_{ij} \theta_\omega \geq \sum_{i \in N} \beta_i + \sum_{k \in K} \zeta_k \qquad ((\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\zeta}) \in R_{\boldsymbol{\Delta}}) \tag{45}$$

$$\sum_{f \in F} \sum_{\omega \in \Omega^f} a_\omega^i \theta_\omega = 1 \qquad (i \in N) \tag{46}$$

$$\sum_{\omega \in \Omega^f} \theta_\omega = 1 \qquad (f \in F) \tag{47}$$

$$\theta_\omega \geq 0 \qquad (f \in F; \omega \in \Omega^f). \tag{48}$$

Feasibility constraints (45) ensure that the values given to the θ_ω variables lead to a bounded dual subproblem. The value of z_0 is then restricted to be larger than or equal to the optimal value of this subproblem by optimality constraints (44). Model (43)-(48) contains more constraints than the LP relaxation of (23)-(30) but most optimality and feasibility constraints are inactive in an optimal solution. Hence, these constraints need not be generated exhaustively. Instead, an iterative algorithm is used to generate only a subset of cuts that are sufficient to identify an optimal solution.

Each iteration of the algorithm solves a relaxed Benders master problem obtained by replacing the sets $P_{\boldsymbol{\Delta}}$ and $R_{\boldsymbol{\Delta}}$ with the subsets $P^t_{\boldsymbol{\Delta}} \subseteq P_{\boldsymbol{\Delta}}$ and $R^t_{\boldsymbol{\Delta}} \subseteq R_{\boldsymbol{\Delta}}$ of extreme points and extreme rays available at iteration $t = 0, 1, \ldots$ The optimal solution of the relaxed Benders master problem is used to set up the primal subproblem (33)-(37). If the primal subproblem is feasible, the values of the dual variables associated with constraints (34)-(36) determine an extreme point of $P_{\boldsymbol{\Delta}}$. Otherwise, an extreme ray of $Q_{\boldsymbol{\Delta}}$ violating inequality (41) is identified. Hence, exactly one constraint is added to the relaxed Benders master problem at each iteration and the process continues until its optimal solution yields a feasible primal subproblem whose cost is equal to the value of z_0.

At each iteration of the Benders decomposition algorithm, the relaxed Benders master problem and the primal subproblem can both be solved by column generation. When the primal subproblem is feasible, the values of the dual variables associated with the constraints of the Dantzig-Wolfe master problem identify an extreme point of $\boldsymbol{\Delta}$ even though only a subset of all columns have been generated. In fact, columns that have not been generated correspond to constraints of the dual subproblem that are already satisfied by the current solution. To avoid generating feasibility constraints from extreme rays in the case when the primal subproblem is infeasible, this subproblem can be made feasible for any solution of the relaxed Benders master problem. This is accomplished by introducing artificial crews, with large positive costs, whose source and sink nodes are connected to every leg node $i \in N$.

Integer solutions to model (23)-(30) can be computed by using the branch-and-bound approach of Section 3.1, solving the linear relaxations by Benders decomposition and column generation. When integrality gaps are not too large, heuristic solutions can also be computed with the following three-phase approach. In the first phase, all integrality requirements are relaxed and the LP relaxation of model (23)-(30) is solved to optimality by Benders decomposition and column generation. Retaining all cuts generated in this first phase, the second phase reintroduces integrality constraints on the master problem variables, and solves the resulting mixed-integer problem by generating additional cuts. In this phase, the integer master problem must be solved at each iteration of the Benders decomposition algorithm. This problem is not necessarily solved to optimality at each iteration because branching is performed directly on path variables as explained at the end of Section 3.1. As a result, the Benders decomposition algorithm may stop with a suboptimal solution even though this solution satisfies the stopping criterion. In the third phase, integrality constraints are finally added on subproblem variables and the integer subproblem is solved once while the values of the master problem variables are held fixed. In this last phase, branching is also performed directly on the path variables. It is worth mentioning that because the subproblem does not have the integrality property, the integer subproblem may be infeasible for the given solution of the master problem even if the original problem is feasible. In practice, however, solutions of very good quality were obtained with this three-phase approach.

4 Computational Experimentation

In this section, we present computational experiments that were carried out on instances based on data provided by a large Canadian airline. We first provide a description of these instances, followed by a summary of our computational experiments.

4.1 Description of data sets

To generate test instances, we started from a weekly schedule, provided by the airline, that contains a total of 3205 short-haul flight legs. Using a fleet assignment algorithm similar to that of Hane et al. (1995), we first assigned an aircraft type to each scheduled flight so as to maximize total anticipated profits. The airline has six different types of aircraft available to cover these flights, but in the fleet assignment solution a total of 2950 flights were covered by just three of these six types. We thus constructed three instances from these flights, discarding the remaining 255 flights that were covered by the other types of aircraft.

To obtain valuable test instances, a certain number of steps were needed to complete the partial data provided by the airline. First, we had to solve a weekly aircraft routing problem to estimate the required fleet size and determine initial positions for the aircraft. This was accomplished by minimizing the objective (1) under constraints (2) and (5)-(9). Because fleet size and initial and final positions were at first unknown, the cardinality of F was set to a large number (e.g., 50) and for each aircraft $f \in F$, we defined $O^f = \{(o^f, i) : i \in N\}$ and $D^f = \{(i, d^f) : i \in N\}$. To each aircraft f, we also assigned an integer m_f, randomly generated from a uniform distribution over the set $\{0, 1, 2, 3\}$, representing the number of days elapsed since the last maintenance. Every aircraft was then required to spend a minimum of 8 hours, at least once every four days, in one of the four cities where maintenance can be performed. An additional arc (o^f, d^f) was also introduced in the set A^f to allow extra aircraft to remain idle during the complete planning horizon. Assigning a large cost to arcs in O^f allowed us to determine a minimum fleet size as well as initial conditions that were used in all further experiments. Because this process makes aircraft with small values of m_f more likely to be used in the solution, each aircraft that covered at least one flight in the solution was assigned a new integer m_f, randomly generated as explained above, to obtained unbiased initial conditions. The aircraft routing problem was then solved again using the reduced fleet size and modified m_f values to verify feasibility. For each of the three instances, the problem remained feasible after performing this adjustment.

The last step in constructing test instances consisted of solving a crew scheduling problem to determine the number of crews necessary as well as their initial positions at the beginning of the planning horizon. To do this, we minimized (1) under constraints (3) and (10)-(14), thus neglecting linking constraints (4). As in the case of aircraft, the cardinality of K was first set to a large number and no specific initial or final conditions were imposed. Each crew was however randomly assigned to one of the four crewbases used by the airline. In addition, each crew k was assigned an integer w_k, chosen randomly in the set $\{0, 1, 2, 3\}$, representing the cumulated work days at the beginning of the planning horizon. When solving the crew scheduling problem, crews are required to return to their base after a maximum of four work days, and can begin a new pairing after spending at least one full day at the base. In addition, the duty period for each work day must respect limits on the total flight time, the total duty time and the total number of landings. Again, assigning a positive cost to arcs in O^k allowed us to determine the required number of crews as well as initial conditions that were used in the subsequent experiments. After solving this problem once, each crew covering at least one flight leg in the solution was then assigned a new random integer w_k and the problem was solved again by retaining only these active crews. Here, additional crews were introduced in each instance to ensure the feasibility of the problem with the linking constraints (4).

Because the set of flight legs to be flown varies significantly from one day to the next in these test instances, solving a daily problem would not be appropriate. Instead, one should either solve a weekly problem or a sequence of shorter subproblems on overlapping time horizons. For test purposes, we thus generated instances with horizons varying between 1 and 3 days. However, when solving a p-day problem, an extended horizon of $p+3$ days is in fact considered: aircraft and crew path constraints (5)-(14) are imposed over the complete horizon whereas covering and linking constraints (2)-(4) are imposed only for the planning horizon, i.e., the first p days. This strategy reduces the effects of the heuristic time horizon decomposition by ensuring the existence of aircraft routings and crew pairings that satisfy maintenance constraints and maximum pairing length constraints, respectively.

The characteristics of the different instances generated are summarized in Table 1. In this table, $|L|$ and $|C|$ correspond to the number of legs and the number of short connections in the first p days of the planning horizon, i.e., those for which constraints (2)-(4) are imposed. Here, a connection between two flight legs l_i and l_j is feasible if the difference between the departure time of leg l_j and the arrival time of leg l_i is greater than or equal to a station-specific threshold t_s that normally varies between 20 and 45 minutes. However, the connection is said to be short if the connection time exceeds t_s by less than 30 minutes. Hence, if the minimum connection time t_s is equal to 30 minutes, all connections whose duration lies between 30 and 60 minutes are said to be short. The numbers in Table 1 also provide the size of model (23)-(30). For example, one can verify that the Dantzig-Wolfe master problem for the largest instance, D9S-3, has a total of $2 \times 525 + 470 + 35 + 67 = 1622$ constraints. For this instance, if model (23)-(30) is solved by Benders decomposition, the relaxed Benders master problem of the first iteration has $525 + 35 = 560$ constraints while the subproblem has $525 + 470 + 67 = 1062$ constraints.

Table 1: Characteristics of test instances

Instance	**Planning horizon (p)**	**Number of legs ($\|L\|$)**	**Number of short connections ($\|C\|$)**	**Number of aircraft ($\|F\|$)**	**Number of crews ($\|K\|$)**
A320-1	1 day	140	93	34	68
A320-2	2 days	273	211	34	68
A320-3	3 days	406	324	34	68
CRJ-1	1 day	128	85	21	51
CRJ-2	2 days	260	183	21	51
CRJ-3	3 days	394	305	21	51
D9S-1	1 day	166	128	35	67
D9S-2	2 days	350	309	35	67
D9S-3	3 days	525	470	35	67

In our experiments, throughs (i.e., sequences of flights to be covered by the same aircraft) were selected when solving the fleet assignment problem. As a result, once fleet size is determined and fixed, the only costs considered in the aircraft routing problem are the operational costs that are associated with the flight legs. But because all legs must be covered exactly once and all aircraft of a given type are assumed to have equal operating costs, this is in fact a fixed cost. The aircraft routing problem thus becomes a feasibility problem to ensure that each aircraft is maintained appropriately. As explained in Section 2, an important portion of all crew costs is also fixed in the sense that each flight must be covered by one crew. Hence, the only relevant costs considered in these experiments are those that can be reduced by a better planning of crew pairings. Expenses are incurred for connections whose duration exceeds a given threshold because crews must then be credited work time even though they are not actually working. Additional accommodation expenses are also incurred when the rest period between successive duties does not take place at the crewbase. Finally, deadhead expenses are incurred when crew members travel as passengers on flights to which they are not assigned to work. Because we were provided only partial information regarding costs, the exact coefficients used in the experiments were parameterized so that variable crew costs represented approximately 5% of total crew costs, a percentage which is close to the industry average for short-haul flights.

4.2 Summary of Computational Experiments

To evaluate the benefits of using the combined Dantzig-Wolfe and Benders decompositions of Section 3.2 as opposed to solving model (23)-(30) directly with column generation only, we first tried to solve each of the nine instances with both approaches. Our algorithms were coded in C and use GENCOL[1] for column generation. All experiments were performed on a Sun Enterprise 10000 (400 MHz) computer, using a single processor.

When solving the problem with Dantzig-Wolfe decomposition, the LP relaxations are solved to optimality but a heuristic branching scheme is used, as explained in Section 3.1. For the combined Dantzig-Wolfe and Benders decomposition, the same branching scheme is used but the LP relaxation is not necessarily solved to optimality. Instead, cuts are generated until the difference between the lower bound provided by the Benders relaxed master problem and the upper bound provided by the subproblem is less than or equal to 1%.

In Table 2, we compare the CPU time needed to solve each of the nine instances with the two approaches. For Dantzig-Wolfe decomposition, column LP reports the time needed to solve the LP relaxation of the Dantzig-Wolfe master problem (23)-(30) whereas column IP indicates the total cumulative time required to compute an integer solution. In the case of the combined Dantzig-Wolfe and Benders decomposition approach, column LP corresponds to the first phase while column IP is the sum of all three phases described in Section 3.2. Because the algorithm is stopped before reaching an optimal solution in the first phase, comparing CPU times for solving the LP relaxation is difficult. However, when considering total CPU times for computing an integer solution, one observes that the combined Dantzig-Wolfe and Benders decomposition approach is considerably faster than the column generation approach and that the difference in performance increases with problem size. For the largest instance (D9S-3), the former approach is more than four times faster than the latter.

Even though a heuristic stopping criterion is used for the combined Dantzig-Wolfe and Benders decomposition approach, the performance improvement does not come at the price of lower solution quality. In Table 2, one can check that the average percentage gap between the cost of the integer solution found and that of the LP relaxation solution

[1]GENCOL is an optimization software that was developed at GERAD in Montreal

Table 2: CPU time and gap statistics for DW and combined DW and Benders decompositions

	DW decomposition			**Combined DW and Benders decompositions**			
	CPU minutes		**Gap**	**CPU minutes**		**Gap**	**Optimality cuts**
Instance	**LP**	**IP**	**(%)**	**LP**	**IP**	**(%)**	**(Phase I / Phase II)**
A320-1	0.51	0.60	0.00	0.44	0.56	0.00	1 / 0
A320-2	3.70	14.53	1.12	2.82	7.41	1.95	4 / 1
A320-3	19.79	62.09	5.78	9.05	23.71	1.78	3 / 0
CRJ-1	1.03	3.34	5.76	1.00	3.24	2.34	3 / 3
CRJ-2	8.72	28.29	4.68	4.50	13.76	1.69	2 / 0
CRJ-3	112.44	377.81	8.61	23.87	106.79	3.42	2 / 2
D9S-1	1.84	7.82	0.94	2.93	5.86	0.97	5 / 1
D9S-2	25.93	93.77	0.66	23.45	63.95	1.80	7 / 1
D9S-3	438.76	1253.98	2.74	64.66	289.36	1.12	3 / 3

is less than 1.7% when using the combined Dantzig-Wolfe and Benders decomposition method. This average gap is smaller than that obtained when solving the problem directly with column generation. Finally, the table reports the number of Benders optimality cuts (constraints (44)) that were generated in the first (I) and second (II) phases of the algorithm. These numbers show that the good performance of the combined approach is in large part explained by the fact that very few cuts are generated in the second phase when the Benders master problem must be solved with integrality constraints.

In the second part of our experiments, we wanted to compare the cost of the solutions obtained with the integrated model (1)-(14) to those obtained with a sequential planning approach in which the aircraft routing problem is solved first and the crew scheduling problem (33)-(37) is solved next, taking the aircraft routing solution as an input. Because the aircraft routing problem is in fact a feasibility problem in which all variables have a cost of 0, it usually has several alternative (optimal) solutions. Hence, two different scenarios were used to compute aircraft routing solutions. In the first scenario (A), penalties were imposed on short connection arcs so as to avoid using them in the solution, thus maximizing the number of linking constraints (35) whose right-hand-side is equal to 0 when solving the crew scheduling problem. This scenario does not necessarily generate the worst aircraft routing but is likely to make most of the linking constraints active. In the second scenario (B), negative penalties were assigned to all short connection arcs so as to make them more attractive when computing the aircraft routing solution and thus minimize the number of linking constraints that are active. In practice, solutions that are close to those of scenario A are often used because they are more robust and less vulnerable to delays.

Table 3 compares the cost of the solutions obtained with the different solution strategies. Columns *Scenario A* and *Scenario B* indicate the cost of the integer solutions obtained with the corresponding sequential planning approach whereas the last column reports the cost of

Table 3: Crew cost comparisons between simultaneous and sequential planning

	Sequential planning		**Simultaneous**
Instance	**Scenario A**	**Scenario B**	**planning**
A320-1	29,902.5	29,005.0	28,528.3
A320-2	56,652.5	56,393.3	52,680.3
A320-3	85,069.2	82,105.8	77,301.7
CRJ-1	23,325.8	22,830.8	21,814.2
CRJ-2	40,794.2	36,314.2	35,755.0
CRJ-3	51,277.5	49,517.5	47,945.8
D9S-1	23,207.5	22,639.2	20,236.7
D9S-2	53,117.5	50,456.7	49,955.0
D9S-3	78,575.0	76,661.7	69,642.5
Total	441,921.7	425,924.2	403,859.5
Daily avg.	73,653.6	70,987.4	67,309.9

the integer solution produced by the combined Dantzig-Wolfe and Benders decomposition approach. These results show that for all instances, the simultaneous planning produced an integer solution of lower cost than those produced by both sequential solution scenarios. The total cost of scenario A and scenario B solutions exceeds that of the simultaneous planning solutions by 9.4% and 5.5%, respectively. The largest relative differences are obtained for instance D9S-1 and are larger than 14.6% for scenario A and 11.8% for scenario B. The last line of Table 3 reports the sum, over all three types of aircraft, of the average daily cost. The average daily cost for an aircraft type was computed by summing the costs of the three solutions for this aircraft type and dividing by 6 (the total number of days in the three instances). From these numbers, one can see that the average daily savings for the complete fleet of 90 aircraft are approximately $6,350 with respect to scenario A and $3,700 with respect to scenario B. These figures translate into annual savings of more than 2.3 and 1.3 million dollars, respectively. For larger airlines with several hundred aircraft, the annual savings would thus represent several million dollars.

Finally, because all relevant costs are in fact crew costs, it may seem reasonable to first determine crew pairings before trying to identify feasible aircraft routings. In the course of our computational experiments, we thus implemented another sequential approach in which the crew scheduling problem was solved first, followed by the aircraft routing problem in which linking constraints were imposed: if a crew made a short connection between legs l_i and l_j, then the same aircraft had to cover both legs. However, this always resulted in an infeasible aircraft routing problem, unless fleet size was increased by a few units. Nevertheless, this alternative approach could possibly be useful when using the combined Dantzig-Wolfe and Benders decompositions.

5 Conclusion

This paper introduces a model and a solution methodology for the simultaneous routing of aircraft and scheduling of crews. The model incorporates aircraft maintenance constraints as well as the most important crew scheduling constraints. These constraints are in fact the basic ones that must be considered by most airlines and may completely represent the work rules of a small regional carrier or those of an airline with a simple collective agreement. It is thus sufficiently realistic to properly evaluate the benefits provided by a better integration of aircraft routing and crew scheduling decisions. On instances based on real data, the simultaneous approach produced important savings with respect to a traditional sequential planning process. It is worth mentioning that the data used in these experiments do not exhibit the popular hub-and-spoke structure. We believe that the potential for savings would be much higher for such networks because of the high number of arrivals and departures that take place within narrow time windows. In this context, there are several crew connections that are feasible only if the same aircraft is used on both legs. Future developments of our approach will address such network structures and will also consider the possibility of having some flexibility on the departure times of the flight legs so as to further reduce crew costs by slightly modifying the flight schedule in a long-term planning context.

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